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CYCLE STRUCTURE IN DISCRETE-DENSITY MODELS

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A super-discrete model is developed in which population densities are positive integers which vary in discrete time intervals. A set of return functions is defined in which each function is a permutation and has a unique maximum value. Phase plane and time plots show much of the complexity of analogous plots for chaotic continuous models with discrete generation times. The analytic description of the cycle structure generated by the set remains an unsolved combinatorial problem.

1. Introduction

In this note we report on our investigations into the cycle structure generated by certain sets of discrete-valued maps. The maps are discrete-valued analogues of first order recurrences of the form $x_{n+1}=f(x_n)$, $n=1,2,\ldots$, where f is a continuous self-map of an interval with the self-regulatory property that x_{n+1} be small for x_n large. Such recurrences have found application in population modelling. Examples include the logistic form $f(x)=rx(1-x)$ defined on $[0,1]$, and the Ricker form $f(x)=x(\exp(r(1-x)))$ defined on $[0,\infty)$. The discrete maps we consider retain this property. We remark that we do not study f's which satisfy only the weaker condition that x_{n+1} be less than x_n for x_n large – as in the Beverton–Holt stock-recruitment relation $x_{n+1}=ax_x/(1+bx_n)$ [1,2].

In the continuous-density case both regular and chaotic cycling can occur in certain subintervals of the parameters [5,6,3,7]. Chaos is meant in the Li-Yorke sense [4], i.e., the recurrence admits all periods $N, N+1, \ldots$, beyond a given one, and there exist aperiodic solutions which are not even asymptotically periodic. There can be no chaos in the classes of discrete maps studied here, essentially because we consider only discrete and finite domains; however, we show that discrete maps can generate complex periodic orbits even when the set of values is relatively small.

Although biomass is thought of as being at least piecewise continuous, there are evidently circumstances in which it is appropriate to consider population numbers. Mathematical chaos is lost, but is replaced by cycle behavior so complex that there is little practical difference. The results illustrate the complex behavior theoretically possible in discrete populations, and anticipate in a simple fashion the dynamics of models with continuous densities.

2. The classes $\Delta(n)$ and $\Lambda(n)$

We define the following set of maps $\Delta(n)$. The map f belongs to $\Delta(n)$ when both the following conditions are satisfied:

(i) f is a permutation of $\{1, 2, \ldots, n\}$.

(ii) f strictly increases up to a unique maximum value and therefore strictly decreases.

A continuous envelope of such a map will be a single-humped curve, the commonly assumed shape for population recurrences with continuous return functions. The value 0 may be added to the domain of f if we assume that $f(0)=0$. The maximum attainable value is n, and there will be some cycle (periodic orbit) generated by f which includes this value. The restrictive condition that $\Delta(n)$ be a set of permutations is relaxed later in studying the containing set $\Lambda(n)$ in which the maps are only required to be *into* $\{1, 2, \ldots, n\}$.

The set $\Delta(3)$ contains two maps which are shown in Fig. 1(a). A graph-theorist would represent the cycles contained in these maps as in Fig. 1(b). The cycles are also indicated in the (x_n, x_{n+1}) coordinate system by the straight-line segments joining successive iterates, using the identity function as a transfer line. The first map decomposes into a 1-cycle (a fixed point) and a 2-cycle, whereas the second map is a 3-cycle. The graph-theoretic representation of a fixed point as in Fig. 1(b) is referred to as a sling.

As defined, $\Delta(n)$ is a subset of the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. As is well known, any permutation decomposes into a finite number of cycles. Starting at 1, there will be some integer k, $1 \leq k \leq n$, such that the mapping sequence $1 \to f(1) \to f^2(1) \to \cdots \to f^k(1)$ terminates at the value $f^k(1)=1$ producing a cycle of length k; if 2 is not in this cycle, then in a similar fashion it will generate its own containing cycle, and so on. (The notation f^k refers to self-composition of f, i.e., $f^2(x)=f(f(x))$ and inductively, $f^n(x)=f(f^{n-1}(x))$, $n=2, 3, \ldots$.) If k_i is the number of cycles of period (or length) which occur in a given map f in $\Delta(n)$, then $1 \cdot k_1 + 2 \cdot k_2 + \cdots + n \cdot k_n = n$.

By induction, the cardinality of $\Delta(n)$ is $2^{n-1}-2$. This compares with $n!$ for the cardinality of all permutations. Maps in $\Delta(n+1)$ are built from those in $\Delta(n)$ essentially by inserting a new maximum value $n+1$ to the immediate left or right of the old one.

A broader classification of maps is the set $\Lambda(n)$, which contains $\Delta(n)$, and whose definition is similar to that of $\Delta(n)$ except that $f \in \Delta(n)$ maps $\{1, 2, \ldots, n\}$ *into* itself. Maps in $\Lambda(n)$ decompose into a finite number of connected *components,* consisting of cycles and numbers which map into these cycles. The graph-theoretic representation of a component is a cycle with attached trees as in Fig. 1(c). This means the only

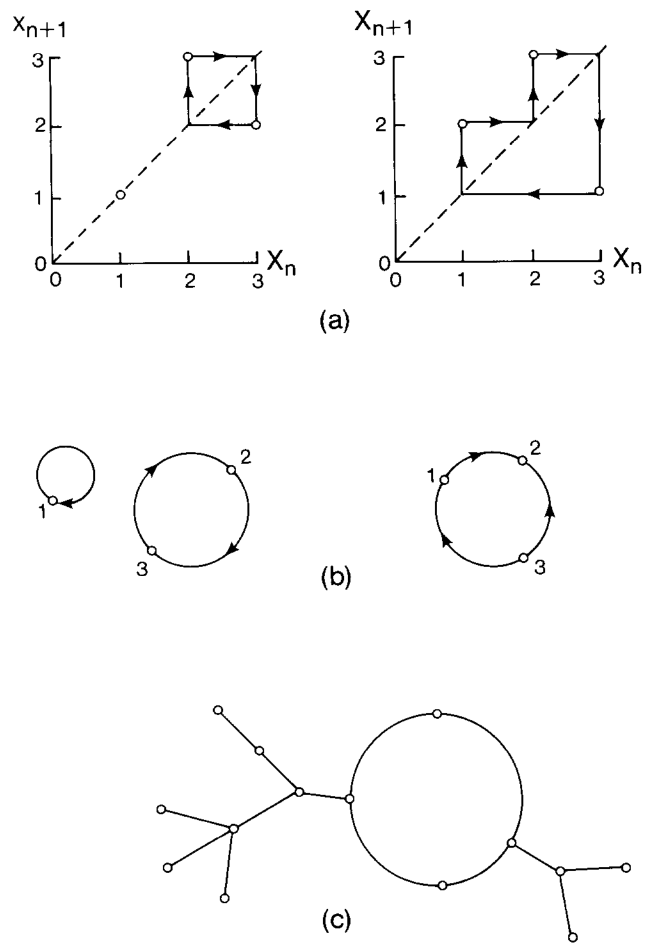

Fig. 1. (a) The elements of $\Delta(3)$, (b) graph-theoretical representation of the elements of $\Delta(3)$, (c) graph-theoretical representation of an element of $\Lambda(n)$. [The attached trees correspond to transient responses.]

interesting feature of the set $\Lambda(n)$ over the contained set $\Delta(n)$ is that an aperiodic trajectory (the complete set of forward iterates under f of a given number) traces out an irregular transient response – the initial part of the trajectory – before finally settling into a periodic cycle. This convergence occurs in finite time, although that time may be long if n is large enough. A segment of a long transient portion may appear chaotic, in the sense of appearing aperiodic. By way of comparison, it is interesting to note that in many first-order recurrences $x_{n+1} = f(x_n)$ with smooth functions f, in the chaotic regime the aperiodic points have measure zero. Also, for n large, the cycles in $\Delta(n)$ and $\Lambda(n)$ will generally have long periods and so are unlikely to be distinguishable from aperiodic trajectories.

The formulas for the cardinality of $\Lambda(n)$ can be found by induction and are

$$\binom{2n-2}{n-1}+\binom{2n-4}{n-1}+\cdots+\binom{n+1}{n-1}+\binom{n-1}{n-1} \quad (n \text{ odd}),$$

$$\binom{2n-2}{n-1}+\binom{2n-4}{n-1}+\cdots+\binom{n+2}{n-1}+\binom{n}{n-1} \quad (n \text{ even}).$$

For n large, an asymptotic formula is

$$\log\|\Lambda(n)\| = 2n\log 2 - \log 3 - \frac{1}{2}\log(\pi n - 1) + o\left(\frac{1}{n}\right)$$

Table 1 compares the cardinalities of the first few sets $\Delta(n)$ and $\Lambda(n)$.

Table 1

n	3	4	5	6	7
$\|\Lambda(n)\|$	5	22	84	312	1161
$\|\Delta(n)\|$	2	6	14	30	62
Number of Cycle Classes in Δ (n)	2	3	6	9	14
Average Number of Classes in Δ (n)	1	2	7/3	10/3	31/7

3. Cycle classes

Here we take the attitude that the added complexity of transient response is spurious, and concentrate on the dynamical properties of the special set of permutations $\Delta(n)$.

Even for modest values of n the maps in $\Delta(n)$ display a high diversity of cyclic behavior. Figs. 2–4 display the 30 maps which belong to $\Delta(6)$. These drawings are in the phase plane (x_n, x_{n+1}). The decomposing cycles of each map are indicated by the line segments joining the successive iterates in each cycle. Fixed points are located on the identity line. Note that any continuous envelope hung over all the points making up the discrete graph will be cap-shaped as required by condition (ii). The corresponding maps in time, x_n vs. n, are shown in Figs. 5–7. Despite the relatively small value of n, there is a glimmer of the rich cycle behavior contained in $\Delta(n)$ when n is large. The maps in the larger set $\Lambda(n)$ will be similar except for the initial, transient-response portions of the graphs; in this case there will be many cycles which are stable for their respective basins of attraction. With respect to a given component, such a basin may be thought of as the union of the attached trees to this component. Thus the dynamical structure of $\Lambda(n)$ differs qualitatively from the usual case in which f is an analytic function and at most one stable cycle exists;

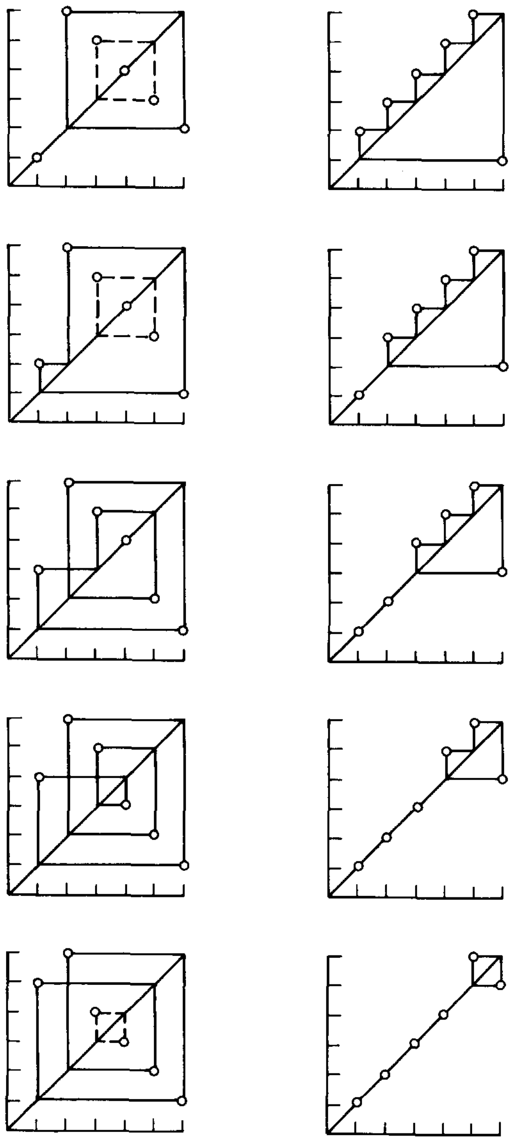

Fig. 2.

Figs. 2–4. Phase-plane representation of the set $\Delta(6)$. The cyclic components of each map are indicated by the joined-line figures. The fixed points are on the diagonal.

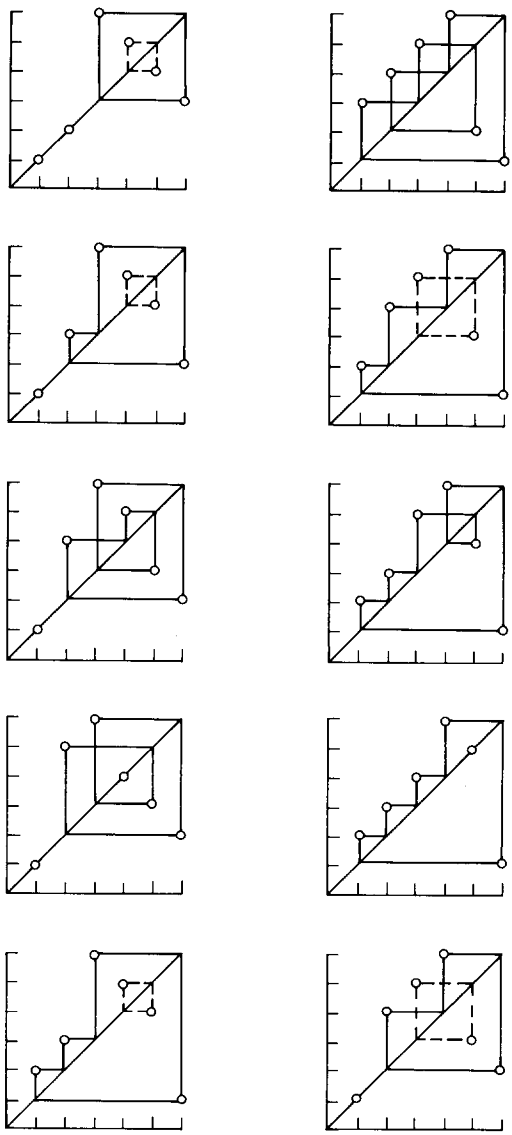

Fig. 3.

actually this uniqueness is lost in general in much nicer classes of maps, e.g., piecewise smooth ones.

We have not been able to compute the cycle classes in $\Delta(n)$; this is apparently a difficult combinatorial problem. As before, let k_i denote the number of cycles of period i which occur in a fixed map, so that $1 \cdot k_1 + 2 \cdot k_2 + \cdots + n \cdot k_n = n$. Define $C(k_1, k_2, \ldots, k_n)$ to be the number of maps in $\Delta(n)$ which have k_1 1-cycles, k_2 2-cycles, ... k_n n-cycles. For example, in $\Delta(3)$, $C(1,1,0)=1$ and $C(0,0,1)=1$. The

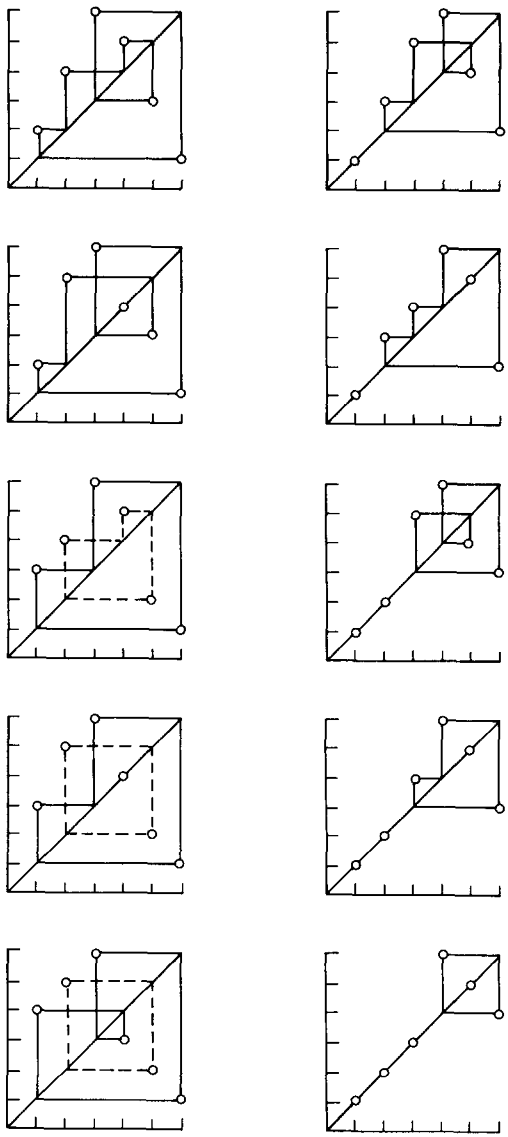

Fig. 4.

question is, for arbitrary n what is the cycle enumerator function C? Riorden (1958) shows that the corresponding question for the set of *all* permutations on n symbols has the answer $n!/1^{k_1}k_1!\,2^{k_2}k_2!\ldots n^{k_n}k_n!$ This provides a poor upper bound on C as restricted to $\Delta(n)$. Even the formula for the number of n-cycles in $\Delta(n)$, i.e., $C(0,\ldots,0,1)$ appears to be technically difficult. Table 2 displays hand-counted formulas for the five smallest sets $\Delta(n)$, $n=3-7$.

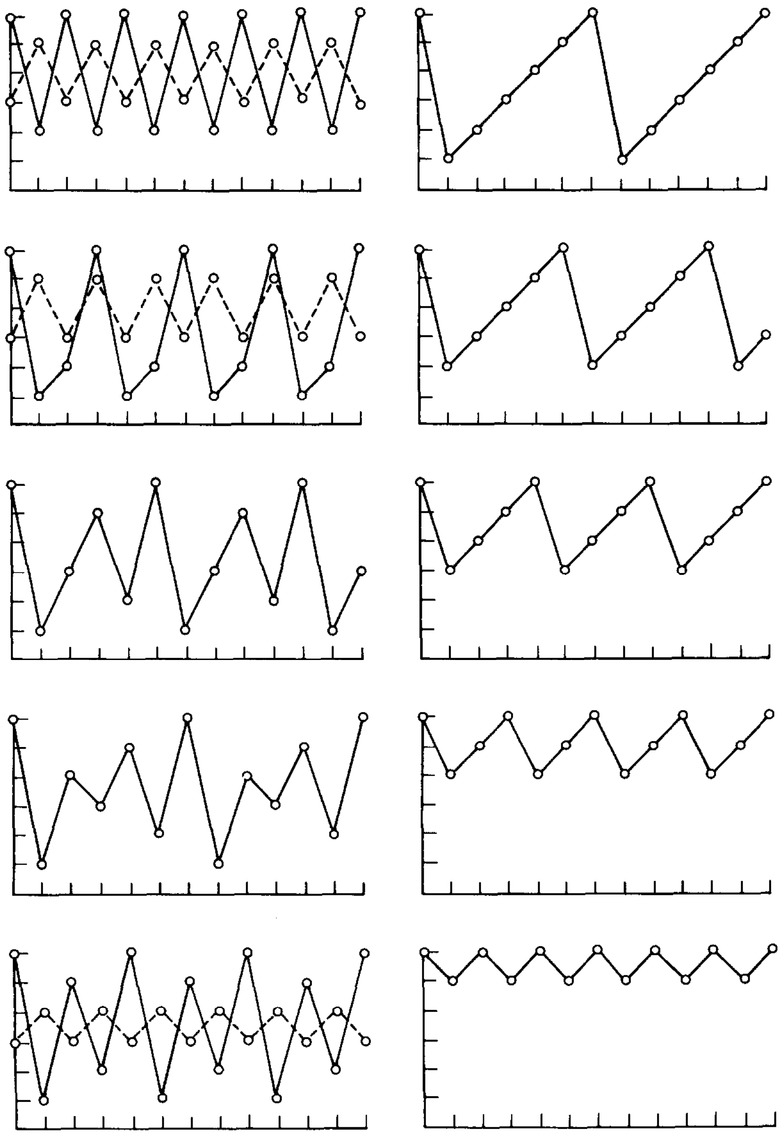

Fig. 5.

Figs. 5–7. Time trajectories of the set $\Delta(6)$. For each map in $\Delta(6)$ only the cycles of period ≥ 2 are shown. For clarity the fixed points of the maps (equilibria) are not drawn. These would appear as horizontal lines in the time domain, had they been drawn.

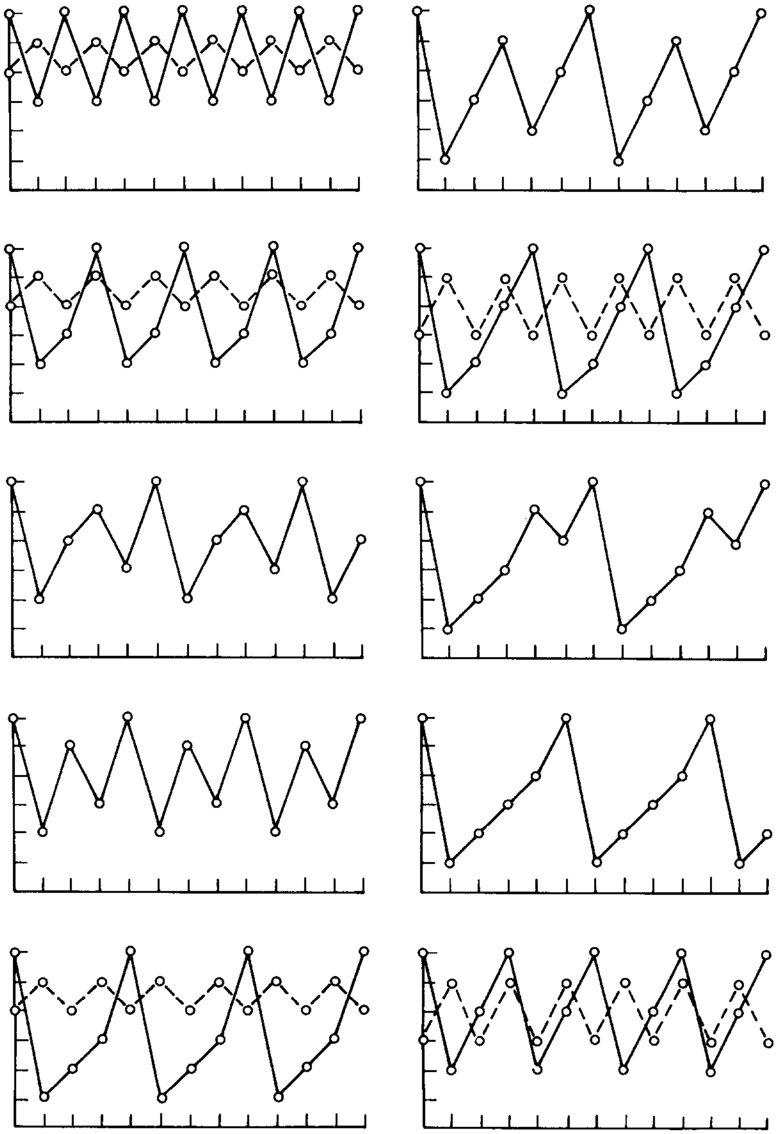

Fig. 6.

4. Remarks

(a) It is clear that discrete-valued recurrences have rich cycle structure, although the upper bound n on the domain of f obviates aperiodic behavior. Cycles in $\Lambda(n)$ are typically preceded in time by a tacked-on transient response. These transients and their attracting cycles will generally be long for large values of n, so that large segments of trajectories will appear chaotic.

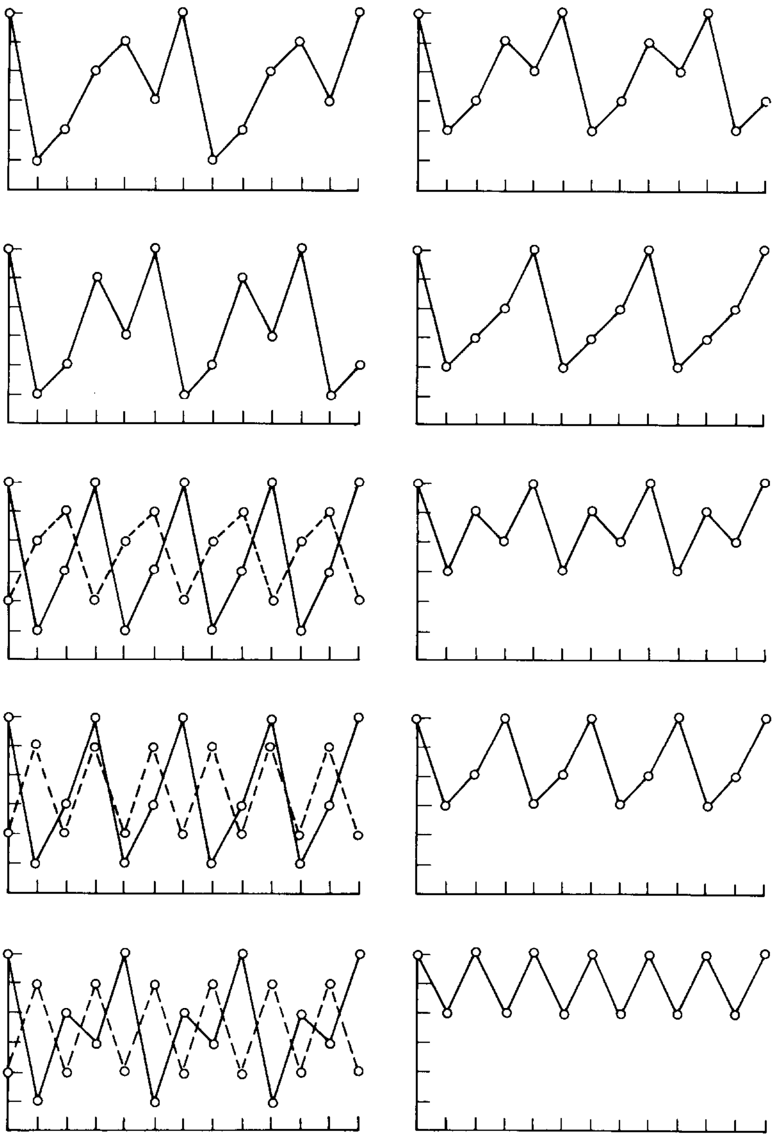

Fig. 7.

(b) As expected, a rescaling of values is possible which permits comparison with the continuous-density case. For example, let $f(i)$, $i = 1, 2, \ldots, n$ denote the set of values of a map in $\Delta(n)$. On the normalized domain $\{1/n, 2/n, \ldots, 1\}$ define a new map g by $g(i/n) = f(i)/n$, $i = 1, 2, \ldots, n$. If i belongs to a k-cycle generated by f, i.e., $f^k(i) = i$, then i/n belongs to a k-cycle generated by g, i.e., $g^k(i/n) = i/n$. The normalized set has identical dynamical structure to $\Delta(n)$. For n large, g closely approxi-

Table 2

Cycle Classes in Δ (n)

Δ (3)	Δ (4)	Δ (5)	Δ (6)	Δ (7)
C(001) = 1	C(0001) = 2	C(00001) = 3	C(000001) = 5	C(0000001) = 9
C(110) = 1	C(1010) = 2	C(01100) = 2	C(010100) = 4	C(1000010) = 11
	C(2100) = 2	C(10010) = 4	C(100010) = 6	C(0100100) = 7
		C(12000) = 1	C(111000) = 4	C(1101000) = 7
		C(20100) = 2	C(002000) = 1	C(0011000) = 4
		C(31000) = 2	C(220000) = 2	C(0210000) = 2
			C(301000) = 2	C(1020000) = 2
			C(410000) = 2	C(1300000) = 1
			C(200100) = 4	C(3200000) = 2
				C(2110000) = 4
				C(2000100) = 5
				C(3001000) = 4
				C(4010000) = 2
				C(5100000) = 2

mates a piecewise-linear curve drawn through the data points. However, we are unable to describe the limiting behavior of such continuous approximations.

It is interesting to note that if the functions are further 'normalized' by converting them to probabilities (accomplished by dividing all the g's by $\frac{1}{2}(n+1)$ because $\sum_{i=1}^{n} g(i/n)/n)/n = \frac{1}{2}(n+1)$), the dynamical structure of $\Delta(n)$ is lost.

The converse problem of attempting to analyze the cycle structure generated by a given continuous map through approximation by members of $\Delta(n)$ appears to be no more transparent because the cycles which decompose the discrete map will not be known a priori. For example, the smooth function $f(x)=4x(1-x)$ on $[0,1]$ produces, upon iteration, periodic cycles which are dense in the interval. However no cycle is attracting and there is a known invariant measure for f, which implies f is ergodic [3]. The denseness property leads us to expect that we may find (normalized) members of $\Delta(n)$ which closely approximate f and contain arbitrary many cycles, but it is difficult to see how any discrete approximations to the ergodic property can arise.

We cannot gloss over the fact that a continuous approximation to a discrete map may not retain any of the original dynamical behavior.

(c) An even more general set of maps, call it $\Omega(n)$ is obtained from $\Delta(n)$ by relaxing the assumption that the ascent to and descent from the maximum value is strictly monotone. Thus $f \in \Omega(n)$ is allowed to have constant portions on the graph, i.e., for $1 \le k \le l \le n$, the relation $f(k)=f(k+1)=\cdots=f(l)$ may hold. Given a cycle of any period and possessing arbitrary values a map in $\Omega(n)$ can be found which has this cycle as one of its components. For if the period is m and the values are $A_1 \le A_2 \le \cdots \le A_m$, then any $f \in \Omega(n)$, $n \le m$ which satisfies $A_{i+1}=f(A_i)$, $i=1,2,\ldots,m-1$, and $A_1=f(A_m)$ will do.

(d) In terms of modelling, it is apparent that even for moderate values of n the cycles have realistic shapes; there can be overtones or beats, i.e., pairs, triples, ... of local maxima which travel through time with the wave. Such structure is characteristic of some experimentally-controlled populations as described in the literature [10].

(e) Any map in $\Delta(n)$ may be considered to be a perturbation of any other (n constant), but such discrete perturbations are discomforting for modelling purposes. There is no built-in mechanism to smoothly perturb one map into another - no tunable parameter. We need to describe how a map can be transformed into a 'nearby' one in the sense of having similar cycle structure.

(f) The domain of the maps in $\Delta(n)$ can be extended to all natural numbers $\{1, 2, \ldots\}$ by specifying that $f \in \Delta(n)$ is zero at $n+1, n+2, \ldots$. Then the union of maps $\bigcup_{n=3}^{\infty} \Delta(n)$ is well-defined, and may be a more appropriate object of study in attempting to concoct a realistic perturbation theory. The sets $\bigcup_{n=3}^{\infty} \Lambda(n)$ and $\bigcup_{n=3}^{\infty} \Omega(n)$ may be analogously defined.

(g) The reader may be interested in referring to Rogers [8] for a similar synthetic and discrete approach involving cell populations, or Rogers and Trofanenko [9] for a study of the evolution of abstract shapes of increasing area.

Acknowledgement

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